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ON THE POSSIBILITY OF SOLVING PLATE STABILITY PROBLEMS WITHOUT A PRELIMINARY DETERMINATION OF THE INITIAL STATE OF STRESS

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Initial stresses in the middle plane enter into the differential equation for plate stability and the corresponding Bryan energy criterion.

In the general case these stresses are determined from the solution of the plane problem. For a plate subjected to complex contour loadings, concentrated forces, for example, the solution of this problem is very complicated.

At the same time, the buckling energy criterion allows a representation in which only the work of the given external forces enters, in addition to the potential energy of plate bending. Hence, the natural question arises as to whether it is generally necessary to know the distribution of the true initial stresses in solving stability problems. It is shown herein that critical values of the external loadings may be found without determining the initial state of stress of the plate.

A new form of the buckling energy criterion is obtained in which the initial stresses do not enter. It is shown that in determining the additional tangential displacements in which the external loadings do work in plate buckling, it is impossible, in the general case, to utilize conditions of inextensibility of the middle plane.

The proposed method of determining the critical loadings without a preliminary solution of the plane problem is illustrated by examples. The known Sommerfeld problem of stability of a rectangular plate compressed by concentrated forces is considered.

1. Let u, v, w be the components of the total displacement vector of points of the middle plane of the plate in a rectangular x, y, z coordinate system. The x and y axes lie in the plane of the plate. The strains in the middle plane of the plate are

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \qquad (1.1)$$

We consider the stresses in the middle plane to satisfy the equilibrium Eqs.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} = 0 \qquad (1.2)$$

and therefore

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 \varphi}{\partial x^2}, \qquad \tau = -\frac{\partial^2 \varphi}{\partial x\, \partial y} \qquad (1.3)$$

The stresses on the plate contour are connected with the loadings X and Y by means of the dependences

$$\sigma_x l + \tau m = X, \qquad \sigma_y m + \tau l = Y \qquad (l = \cos\alpha,\ m = \sin\alpha) \tag{1.4}$$

Here α is the angle formed by the external normal to the contour of the plate and the x-axis. On the contour correspondingly

$$\frac{\partial\varphi}{\partial x} = \int_0^s X\,ds, \qquad \frac{\partial\varphi}{\partial y} = -\int_0^s Y\,ds \tag{1.5}$$

Hooke's law for the stresses and strains in the middle plane is

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y), \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x), \quad \gamma = \frac{2(1+\mu)}{E}\tau \tag{1.6}$$

where E and μ are the elastic modulus and Poisson's ratio of the plate material.

At the time preceding buckling let

$$w = 0, \qquad v = v_0, \qquad u = u_0, \qquad \varphi = \varphi_0, \qquad \varepsilon_x = \varepsilon_{x0}, \ldots \tag{1.7}$$

After buckling let

$$v = v_0 + v_1, \qquad u = u_0 + u_1, \qquad \varphi = \varphi_0 + \varphi_1,\ \varepsilon_x = \varepsilon_{x0} + \varepsilon_{x1}, \ldots \tag{1.8}$$

The stress function φ_0 satisfies the biharmonic Eq. $\nabla^2\nabla^2\varphi_0 = 0$. The stress function φ_1 satisfies the known Karman Eq.

$$\nabla^2\nabla^2\varphi_1 = E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{1.9}$$

The linearized equilibrium Eq. of a curved element of the plate in normal projections is

$$D\nabla^2\nabla^2 w = h\left(\sigma_{x0}\frac{\partial^2 w}{\partial x^2} + \sigma_{y0}\frac{\partial^2 w}{\partial y^2} + 2\tau_0\frac{\partial^2 w}{\partial x\,\partial y}\right) \qquad \left(D = \frac{Eh^3}{12(1-\mu^2)}\right) \tag{1.10}$$

where h is the plate thickness.

The known Bryan energy criterion [1 and 2] for absolute equilibrium during plate buckling is

$$V + \frac{h}{2}\iint\left[\sigma_{x0}\left(\frac{\partial w}{\partial x}\right)^2 + \sigma_{y0}\left(\frac{\partial w}{\partial y}\right)^2 + 2\tau_0\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dx\,dy = 0 \tag{1.11}$$

where V is the potential energy of plate bending

$$V = \frac{D}{2}\iint\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\mu)\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\}dx\,dy \tag{1.12}$$

The stresses $\sigma_{x0}, \sigma_{y0}, \tau_0$ should be determined from the solution of the plane problem of elasticity theory. Let us use the notation

$$U = \frac{h}{2}\iint\left[\sigma_{x0}\left(\frac{\partial w}{\partial x}\right)^2 + \sigma_{y0}\left(\frac{\partial w}{\partial y}\right)^2 + 2\tau_0\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dx\,dy \tag{1.13}$$

2. The derivatives of the transverse deflection w in (1.13) may be expressed in terms of the additional displacements u_1, v_1 and the additional stresses $\sigma_{x1}, \sigma_{y1}, \tau_1$. It follows from (1.1) and (1.6) that

$$\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \varepsilon_{x1} - \frac{\partial u_1}{\partial x} = \frac{1}{E}(\sigma_{x1} - \mu\sigma_{y1}) - \frac{\partial u_1}{\partial x}$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \varepsilon_{y1} - \frac{\partial v_1}{\partial y} = \frac{1}{E}(\sigma_{y1} - \mu\sigma_{x1}) - \frac{\partial v_1}{\partial y} \tag{2.1}$$

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \gamma_1 - \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) = \frac{2(1+\mu)}{E}\tau_1 - \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)$$

On the basis of (2.1) and the dependences

$$\frac{1}{E}(\sigma_{x0} - \mu\sigma_{y0}) = \frac{\partial u_0}{\partial x}, \quad \frac{1}{E}(\sigma_{y0} - \mu\sigma_{x0}) = \frac{\partial v_0}{\partial y}, \quad \frac{2(1+\mu)}{E}\tau_0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \tag{2.2}$$

for the initial state of stress, (1.13) may be transformed to

$$U = h\iint\left[\frac{\partial u_0}{\partial x}\sigma_{x1} + \frac{\partial v_0}{\partial y}\sigma_{y1} + \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)\tau_1\right]dx\,dy - h\iint\left[\frac{\partial u_1}{\partial x}\sigma_{x0} + \right.$$

$$\left. + \frac{\partial v_1}{\partial y}\sigma_{y0} + \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)\tau_0\right]dx\,dy \tag{2.3}$$

Utilizing the Green's formula of integration by parts, we obtain

$$U = h \oint [u_0\sigma_{x1}l + v_0\sigma_{y1}m + u_0\tau_1 m + v_0\tau_1 l]\, ds - h \iint \Big[u_0 \left(\frac{\partial \sigma_{x1}}{\partial x} + \frac{\partial \tau_1}{\partial y} \right) +$$

$$+ v_0 \left(\frac{\partial \sigma_{y1}}{\partial y} + \frac{\partial \tau_1}{\partial x} \right) \Big]\, dx\, dy - h \oint [u_1\sigma_{x0}l + v_1\sigma_{y0}m + u_1\tau_0 m + v_1\tau_0 l]\, ds +$$

$$+ h \iint \Big[u_1 \left(\frac{\partial \sigma_{x0}}{\partial x} + \frac{\partial \tau_0}{\partial y} \right) + v_1 \left(\frac{\partial \sigma_{y0}}{\partial y} + \frac{\partial \tau_0}{\partial x} \right) \Big]\, dx\, dy \tag{2.4}$$

By virtue of conditions (1.2), which are satisfied for both the initial and the tangential stresses, the double integrals vanish. The factors in the displacements u_0, v_0 and u_1, v_1 in the integrands of the contour integrals transform to contour loadings according to (1.4). Therefore, the energy criterion for indifferent equilibrium during buckling may be written as

$$V + h \oint [u_0 X_1 + v_0 Y_1]\, ds - h \oint [u_1 X_0 + v_1 Y_0]\, ds = 0 \tag{2.5}$$

Here X_0, Y_0 and u_0, v_0 are initial contour loadings and displacements, X_1, Y_1 and u_1, v_1 are additional contour loadings and the corresponding displacements.

If the forces X_0, Y_0 are given on the plate contour (first boundary value problem), the additional contour forces X_1, Y_1 are equal to zero, and the buckling energy criterion (2.5) becomes

$$V - h \oint [u_1 X_0 + v_1 Y_0]\, ds = 0 \tag{2.6}$$

If the displacements u_0, v_0 are given on the plate contour (second boundary value problem), then the additional displacements u_1, v_1 vanish on the contour and we obtain in place of condition (2.6)

$$V + h \oint [u_0 X_1 + v_0 Y_1]\, ds = 0 \tag{2.7}$$

For the first problem the boundary conditions $X_1 = Y_1 = 0$ may also be written thus

$$\frac{\partial \varphi_1}{\partial x} = \frac{\partial \varphi_1}{\partial y} = 0 \qquad \text{or} \qquad \varphi_1 = \frac{\partial \varphi_1}{\partial n} = 0 \tag{2.8}$$

Eq. (2.6) may be considered as the buckling energy criterion in the form of Timoshenko.

3. Instead of the actual initial stresses σ_{x0}, σ_{y0}, τ_0 let us introduce the statically admissible stresses σ_{x0}^+, σ_{y0}^+, τ_0^+. They satisfy the equilibrium Eqs. (1.2) and the boundary conditions (1.4), but cannot satisfy the equation of compatibility of the strains $\nabla^2(\sigma_x + \sigma_y) = 0$. For any system of statically admissible stresses, we may write the identity

$$\oint [u_1 X_0 + v_1 Y_0]\, ds = \iint \left[\frac{\partial u_1}{\partial x} \sigma_{x0}^+ + \frac{\partial v_1}{\partial y} \sigma_{y0}^+ + \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) \tau_0^+ \right] dx\, dy \tag{3.1}$$

Therefore, the Timoshenko's energy criterion (2.6) may be written as

$$V - h \iint \left[\frac{\partial u_1}{\partial x} \sigma_{x0}^+ + \frac{\partial v_1}{\partial x} \sigma_{y0}^+ + \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) \tau_0^+ \right] dx\, dy = 0 \tag{3.2}$$

Utilizing (2.1) to eliminate the derivatives of the displacements u_1 and v_1 we obtain

$$V + \frac{h}{2} \iint \left[\sigma_{x0}^+ \left(\frac{\partial w}{\partial x} \right)^2 + \sigma_{y0}^+ \left(\frac{\partial w}{\partial y} \right)^2 + 2\tau_0^+ \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx\, dy - \tag{3.3}$$

$$- \frac{h}{E} \iint (\sigma_{x0}^+ + \sigma_{y0}^+)(\sigma_{x1} + \sigma_{y1})\, dx\, dy + \frac{h(1+\mu)}{E} \iint (\sigma_{x0}^+ \sigma_{y1} - 2\tau_0^+ \tau_1 + \sigma_{y0}^+ \sigma_{x1})\, dx\, dy = 0$$

The last integral on the left-hand side of (3.3) vanishes. This is easily proved by replacing σ_{x1}, σ_{y1}, τ_1 by their expressions in terms of derivatives of the Airy function φ_1 and integrating by parts. The contour integral drops out by virtue of the boundary conditions (2.8), and the integral over the area drops out by virtue of the equilibrium Eqs. (1.2) for the stresses σ_{x0}^+, σ_{y0}^+, τ_0^+.

Therefore, the energy Eq. (2.6) may be given the form

$$V + \frac{h}{2} \iint \left[\sigma_{x0}^+ \left(\frac{\partial w}{\partial x} \right)^2 + \sigma_{y0}^+ \left(\frac{\partial w}{\partial y} \right)^2 + 2\tau_0^+ \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx\, dy -$$

$$-\frac{h}{E}\iint(\sigma_{x0}^{+}+\sigma_{y0}^{+})(\sigma_{x1}+\sigma_{y1})\,dx\,dy=0 \tag{3.4}$$

If the statically admissible stresses coincide with the true stresses, then

$$\iint(\sigma_{x0}+\sigma_{y0})(\sigma_{x1}+\sigma_{y1})\,dx\,dy=\iint\nabla^2\varphi_0\nabla^2\varphi_1 dx\,dy=\iint(\nabla^2\nabla^2\varphi_0)\,\varphi_1 dx\,dy=0 \tag{3.5}$$

and (3.4) transforms into the original Eq. (1.11). Eq. (3.4) is a new energy criterion for stability, expressed in terms of the statically admissible stresses $\sigma_{x0}^{+}, \sigma_{y0}^{+}, \tau_0^{+}$. Upon using this criterion it is not required to solve the plane problem of elasticity theory to determine the initial stresses $\sigma_{x0}, \sigma_{y0}, \tau_0$. The unknown function φ_1 should be determined from the solution of the Karman Eq. (1.9) under the boundary conditions (2.8) for a given function of the transverse deflection w.

The necessity of solving the nonlinear Eq. (1.9) seems, at first glance, to be peculiar In order to clarify the singularities of the obtained problem, let us consider the equations for u_1 and v_1. From (1.1) and (1.6) we have:

$$\begin{aligned}
\frac{\partial u_1}{\partial x}&=-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{E}\left(\frac{\partial^2\varphi_1}{\partial y^2}-\mu\frac{\partial^2\varphi_1}{\partial x^2}\right)\\
\frac{\partial v_1}{\partial y}&=-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{1}{E}\left(\frac{\partial^2\varphi_1}{\partial x^2}-\mu\frac{\partial^2\varphi_1}{\partial y^2}\right)\\
\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}&=-\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-\frac{2(1+\mu)}{E}\frac{\partial^2\varphi_1}{\partial x\,\partial y}
\end{aligned} \tag{3.6}$$

From a consideration of (3.6) and (1.9) it follows that the second members on the right-hand sides of these Eqs., defined by the additional Airy function are of the same order of smallness as the first members which are expressed in terms of squares and products of the derivatives of the transverse deflection of the plate.

In a number of works on plate stability, starting with the early works of Timoshenko, the displacements u_1 and v_1 on the plate contour were found from the so-called conditions of inextensibility of the middle plane of the plate

$$\frac{\partial u_1}{\partial x}=-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\qquad \frac{\partial v_1}{\partial y}=-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\qquad \frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}=-\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{3.7}$$

However, the system of Eqs. (3.7) will be compatible if and only if

$$\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}=0 \tag{3.8}$$

i.e., when the right-hand side of (1.9) vanishes and $\varphi_1(x, y)=0$. But condition (3.8) is satisfied only for bending of the plate along a developable surface, for example, in the cylindrical bending case. Formal detemination of the displacements $u_1(x, y)$ or $v_1(x, y)$ on the plate contour by integrating one of the first two Eqs. of the system (3.7) is not legitimate if (3.8) is not satisfied.

It is interesting to note that the Bryan Eq. (1.11) can be obtained from the energy condition (2.6) by formally utilizing (3.7). This equation was obtained precisely thus by Timoshenko. However, in the general case it is impossible to obtain correct values of the displacements $u_1(x, y)$ and $v_1(x, y)$ from (3.7). Let us also note an interesting property of the true stresses $\sigma_{x0}, \sigma_{y0}, \tau_0$, which differentiates them from the statically admissible stresses $\sigma_{x0}^{+}, \sigma_{y0}^{+}, \tau_0^{+}$. By subtracting (2.6) from (1.11) we obtain

$$\iint\left[\sigma_{x0}\left(\frac{\partial w}{\partial x}\right)^2+\sigma_{y0}\left(\frac{\partial w}{\partial y}\right)^2+2\tau_0\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dx\,dy+\oint[u_1X_0+v_1Y_0]\,ds=0$$

Replacing the contour integral by an area integral and taking account of (1.11), we obtain

$$\iint(\sigma_{x0}\varepsilon_{x1}+\sigma_{y0}\varepsilon_{y1}+\tau_0\gamma_1)\,dx\,dy=0 \tag{3.9}$$

The relationship (3.9) may be considered as an original orthogonality condition for the true initial stresses and the additional strains, occurring at the time of buckling, of the middle plane of the plate.

Condition (3.9) is not satisfied for arbitrary statically admissible initial stresses. The orthogonality condition (3.9) may also be written as

$$\iint \nabla^2\varphi_0\nabla^2\varphi_1\, dx\, dy = 0 \tag{3.10}$$

Letting $\varphi_0{}^+$ denote the Airy function for statically admissible initial stresses, we obrain from (3.4)

$$V + \frac{h}{2}\iint\left[\sigma_{x0}{}^+\left(\frac{\partial w}{\partial x}\right)^2 + \sigma_{y0}{}^+\left(\frac{\partial w}{\partial y}\right)^2 + 2\tau_0{}^+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dx\, dy - \\ - \frac{h}{E}\iint \nabla^2\varphi_0{}^+\nabla^2\varphi_1 dx\, dy = 0 \tag{3.11}$$

Eq. (1.9) and the boundary conditions (2.8) should lead to this equation.

The scheme for solving the problem of determining the critical values of external loadings applied to a plate contour in its plane is the following. For given external loadings on the plate contour the simplest statically admissible system of stresses $\sigma_{x0}{}^+$, $\sigma_{y0}{}^+$, $\tau_0{}^+$ is determined. The transverse deflection of the plate is given as the linear aggregate

$$w = \Sigma f_w w_m(x,\ y) \tag{3.12}$$

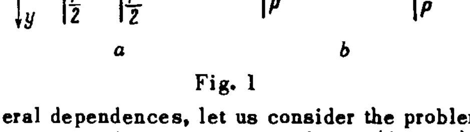
as is done in solving problems by the Ritz method.

The right-hand side of (1.9) is evaluated, and it is solved either exactly or approximately for homogeneous boundary conditions (2.8) on the contour. The solution of this auxiliary problem is analogous to the solution of the problem of transverse bending of a plate clamped along the contour. Having determined $\varphi_1(x,\ y)$ and $\nabla^2\varphi_1(x,\ y)$ the last member of the left-hand side of (3.11) can be evaluated and the ordinary procedure of the Ritz-Timoshenko method may then be applied to determine the critical values of the parameters governing the external contour loadings X_0, Y_0. The original Eq. (2.6) of the Timoshenko method may be considered in place of (3.11). But the displacements u_1 and v_1 on the plate contour should be determined by means of integrating the correct Eqs. (3.6), and not from the erroneous Eqs. (3.7).

Therefore, utilization of the Timoshenko energy method in the form of (2.6) does not require preliminary determination of the actual stress field in the plate prior to buckling, but its correct application requires solution of the Karman Eq. (1.9) and the determination of the tangential displacements taking account of the additional stresses σ_{x1}, σ_{y1}, τ_1 occurring during plate buckling.

In many cases, particularly for a plate subjected to concentrated forces, the solution of (1.9) under the simple boundary conditions (2.8) is much simpler than the solution of the biharmonic Eq. $\nabla^2\nabla^2\varphi_0 = 0$ under complex boundary conditions.

The energy Eq. (2.6) may be considered as a corollary of the energy conservation law in the bifurcation of the plate equilibrium mode. But an apparent contradiction arises in such an analysis of this equation and the results obtained in this paper. It follows from (2.6) that all the work of the external forces on the additional displacements transforms into bending potential energy during plate buckling. And it follows from (3.6) that it is necessary to take account of the additional stresses in the middle plane of the plate. This contradiction is easily explained if it is noted that the potential energy of these additional stresses will be a higher order quantity. But these stresses must be taken into account in the determination of the tangential displacements of points of the middle plane.



a b

Fig. 1

4. As an illustration of applying the general dependences, let us consider the problem of stability of a freely supported square plate compressed by concentrated forces (Fig. 1*a*). The boundary conditions for $w(x,\ y)$ are:

$$w = 0,\quad \partial^2 w/\partial x^2 = 0 \quad \text{for} \quad x = 0,\ x = a;$$
$$w = 0,\quad \partial^2 w/\partial y^2 = 0 \quad \text{for} \quad y =)\qquad y = a \tag{4.1}$$

The boundary conditions (2.8) are valid for the function $\varphi_1(x,\ y)$.

Let us give approximate expressions for w and φ_1 in the form

$$w = f_1 \sin\frac{\pi x}{a}\sin\frac{\pi y}{a}, \qquad \varphi_1 = c_1 \sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a} \tag{4.2}$$

Integrating (1.9) by the Galerkin method, we find $c_1 = -f_1{}^2E/4$. According to (1.12) we obtain $V = \pi^2 f_1 D/2a^2$. By virtue of the symmetry of the problem only half the plate for $x \leqslant a/2$ may henceforth be considered. The statically admissible stresses are selected as follows:

$$\sigma_{y0}{}^+ = -\frac{P}{2h\varepsilon} \quad \text{for} \quad \left(c - \frac{\varepsilon}{2}\right) < x < \left(c + \frac{\varepsilon}{2}\right)$$
$$\sigma_{y0}{}^+ = 0 \quad \text{for} \quad 0 \leqslant x \leqslant \varepsilon/2,\ (c + \varepsilon/2) \leqslant x \leqslant a/2 \tag{4.3}$$
$$\sigma_{x0}{}^+ = 0, \quad \tau_0{}^+ = 0 \quad (\varepsilon \to 0)$$

Then taking into account that

$$\nabla^2\varphi_1 = C_1\left(\frac{\pi}{a}\right)^2\left(\cos\frac{2\pi x}{a} - 2\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a} + \cos\frac{2\pi y}{a}\right) \tag{4.4}$$

we obtain from (3.7)

$$\frac{\pi^4}{a^2}\frac{f_1{}^2D}{2} - \frac{\pi^2 f_1{}^2 P_*}{4a}\sin\frac{2\pi c}{a} - \frac{\pi^2 f_1{}^2 P_*}{16a}\cos\frac{2\pi c}{a} = 0 \tag{4.5}$$

where P_* is the critical force. Hence

$$\bar{P}_* = \frac{P_* a}{4\pi^2 D} = \frac{1}{1 - {}^1/_2\cos 2\pi c/a} \tag{4.6}$$

In particular, we obtain $\bar{P}_* = 2/3$ and $\bar{P}_* = 2$, respectively, for $c = a/2$ and $c = 0$.
Using (3.6), we obtain on the plate contour

$$\bar{v}_1 = \frac{8av_1}{\pi^2 f_1{}^2} = 1 - \frac{1}{2}\cos\frac{2\pi x}{a} \quad \text{for} \quad y = 0 \tag{4.7}$$

and we again arrive at (4.6) from the condition (2.6).

The problem of stability of a rectangular plate loaded by concentrated forces has an interesting history. It was first considered by Sommerfeld [3] in 1906; then Timoshenko [1], Filippov [4], Lur'e [5] and others solved it in several different variants and for different boundary conditions. All these authors obtained the value $\bar{P}_* = 0.478$, substantially below the value $\bar{P}_* = 2/3 \approx 0.667$ given by (4.6), for a square hinge supported plate compressed by two concentrated forces (for $c = a/2$ in the problem considered above).

Most recently, several new works have appeared where this same problem has been solved by utilizing computers. The field of initial real stresses has been determined numerically in these works, and then the stability problem has also been solved numerically. In particular, $\bar{P}_* = 0.650$ [6] and $\bar{P}_* = 0.675$ [7] have been obtained for a square hinge supported plate (with $c = a/2$) by such a method.

The value $\bar{P}_* = 0.478$, obtained previously, is explained by the fact that the statically admissible field of initial stresses (4.3) was used in place of the real field of initial stresses, and the determination of the critical loadings was actually carried out by Formula (3.11) but without taking account of the last integral.

If we try to solve the problem considered above by using the condition of inextensibility of the middle plane, and to determine $v_1(x, y)$ from just the second Eq. of the system (3.7), we then obtain in place of (4.7) and (4.6)

$$\bar{v}_1 = 1 - \cos\frac{2\pi x}{a}, \qquad \bar{P}_* = \frac{1}{1 - \cos 2\pi c/a} \tag{4.8}$$

For $c = a/2$ we obtain $\bar{P}_* = 0.5$, and for $c \to 0$ we arrive at the absurd result $\bar{P}_* \to \infty$. This very same result $\bar{P}_* \to \infty$ is also obtained in solving the problem by (3.11) if the last integral in it is not taken into account.

An analogous example (Fig. 1*b*), when the condition of inextensibility of the middle plane leads to infinite critical loadings, was presented in [8]. But the authors of [8] made the erroneous deduction from this clever example that it is necessary to determine the true initial stresses in the plate to solve this problem. Using the approximate dependences (4.2), (4.3) and (4.5) obtained earlier, it is easy to find that $\bar{P}_* = 1/2$ without determining the initial stresses.

Let us note that the obtained results are easily extended to the problem of stability and oscillations of plates loaded by mass forces, and in particular, to the problem of the natural oscillations of a rotating disc stretched by centrifugal forces.

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